



Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at <http://about.jstor.org/participate-jstor/individuals/early-journal-content>.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

$$\begin{aligned}
\therefore \Delta &= 16 \int_0^{\frac{1}{2}\pi} \int_0^{b'} \int_0^{b \sec \theta} \sqrt{(r^2 - \rho^2)} \rho d\theta db d\rho / \int_0^{b'} db \\
&= \frac{16\sqrt{2}}{3r} \int_0^{\frac{1}{2}\pi} \int_0^{b'} [r^3 - (r^2 - b^2 \sec^2 \theta)^{\frac{3}{2}}] d\theta db \\
&= \frac{r^3\sqrt{2}}{3} \int_0^{\frac{1}{2}\pi} \left[8\sqrt{2} - (5 - \sec^2 \theta)\sqrt{(2 - \sec^2 \theta)} - 6\cos\theta \sin^{-1} \left(\frac{\sec \theta}{\sqrt{2}} \right) \right] d\theta \\
&= \frac{r^3\sqrt{2}}{3} \left[2\pi\sqrt{2} - \frac{3}{2}\pi\sqrt{2} - \int_0^{\frac{1}{2}\pi} \left((5 - \sec^2 \theta)\sqrt{(2 - \sec^2 \theta)} - \frac{6\tan^2 \theta}{\sqrt{(2 - \sec^2 \theta)}} \right) d\theta \right].
\end{aligned}$$

Let $\tan \theta = \sin \phi$.

$$\begin{aligned}
\therefore \Delta &= \frac{r^3\sqrt{2}}{3} \left[\frac{1}{2}\pi\sqrt{2} - \int_0^{\frac{1}{2}\pi} \frac{4\cos^2 \phi - \sin^2 \phi \cos^2 \phi - 6\sin^2 \phi}{1 + \sin^2 \phi} d\phi \right] \\
&= \frac{r^3\sqrt{2}}{3} \left[\frac{1}{2}\pi\sqrt{2} - \int_0^{\frac{1}{2}\pi} \left(\sin^2 \phi - 12 + \frac{16}{1 + \sin^2 \phi} \right) d\phi \right] = \frac{\pi r^3}{12} (23\sqrt{2} - 28).
\end{aligned}$$

Also solved by the Proposer.

GROUP THEORY.

2. Proposed by W. BURNSIDE, The Croft, Bromley Road, Catford, England.

Show that the group of the biquadratic equation $x^4 + 2ax^2 + b = 0$, in which a and b are rational numbers, while $a^2 - b$ is not the square of a rational number, is in general a dihedron group of order 8; but that (i) if b is the square of a rational number the group is a non-cyclical Abelian group of order 4; and (ii) if $b = a^2 \div (1 + n^2)$, where n is a rational number, the group is a cyclical group of order 4.

Solution by L. E. DICKSON, Ph. D., The University of Chicago.

The problem may be solved most readily by making use of the simple expressions resulting for the roots of this special quartic. To obtain a more instructive example of the application of groups to equations, I treat the problem without assuming the quartic to be solvable by radicals.

Consider the general quartic $x^4 + ax^3 + bx^2 + cx + d = 0$, and call its roots x_1, x_2, x_3, x_4 . As is well known, the functions

$$y_1 = x_1x_2 + x_3x_4, \quad y_2 = x_1x_3 + x_2x_4, \quad y_3 = x_1x_4 + x_2x_3$$

are the roots of the cubic resolvent

$$y^3 - by^2 + (ac - 4d)y - a^2d + 4bd - c^2 = 0.$$

For our special quartic, this resolvent becomes

$$y^3 - 2ay^2 - 4by + 8ab \equiv (y - 2a)(y^2 - 4b) = 0.$$

We may therefore, by fixing the notation of the x 's, set $y_1 = 2a$, $y_2 = 2\sqrt{b}$, $y_3 = -2\sqrt{b}$. These three values are distinct since $a^2 - b$ is not the square of a rational number by hypothesis. Further, y_1 has a rational value. Hence* the group G of the quartic is contained in the dihedron group G_8 ,

$$I, (12), (34), (12)(34), (13)(24), (14)(23), (1324), (1423),$$

which leaves $x_1x_2 + x_3x_4$ formally unaltered. The only subgroups of order 4 of G_8 are the cyclic C_4 , generated by (1324), and

$$G_4 = [I, (12), (34), (12)(34)], \quad D_4 = [I, (12)(34), (13)(24), (14)(23)].$$

Now the desired group G is contained in D_4 if and only if b is the square of a rational number. Indeed, $x_1x_3 + x_2x_4 = 2\sqrt{b}$, from above, and $x_1x_3 \cdot x_2x_4 = b$, from the quartic, so that $x_1x_3 = x_2x_4 = \sqrt{b}$. Since x_1x_3 is thus numerically unaltered by the substitutions of D_4 , it follows that G does not lie in D_4 if \sqrt{b} is irrational. That G lies in D_4 if \sqrt{b} is rational follows since the rational relation $x_1x_3 = \sqrt{b}$ does not remain true upon applying (12), (34), (1324), or (1423), the quartic having no double root.

Next, G is not contained in G_4 , which leaves x_1x_2 formally invariant. Indeed, $x_1x_2 + x_3x_4 = 2a$ and $x_1x_2 \cdot x_3x_4 = b$ give $x_1x_2 = a \pm \sqrt{a^2 - b}$, which is irrational.

Finally, G is contained in C_4 if and only if $(a^2 - b)/b$ is the square of a rational number n , whence $b = a^2/(4n^2)$. The proof is simplified by the remark that the quartic contains only even powers of x so that two of its roots are the negatives of two others. In view of the values of the y_i , we may set $x_2 = -x_1$, $x_4 = -x_3$. Then $x_1^2 + x_3^2 = -y_1 = -2a$. But $x_1x_3 = \sqrt{b}$. Hence $x_1^2 - x_3^2 = 2\sqrt{a^2 - b}$. Then

$$f \equiv \frac{x_1}{x_3} - \frac{x_3}{x_1} = 2\sqrt{\frac{a^2 - b}{b}}.$$

As above, f is rational only when $b = a^2/(1 + n^2)$. Now f is numerically unaltered by (1324), but changed into its negative by (12), (34), (13)(24), or (14)(23).

To insure that G shall not reduce to a group of order 2, necessarily generated by (12)(34), (13)(24), or (14)(23), we add a condition to insure that the quartic shall be irreducible so that its group will be transitive. Since $a^2 - b$ is not the square of a rational number, the quartic $(x^2 + a)^2 = a^2 - b$ has no linear factor. It will have the factors $x^2 \pm cx + d$ if and only if $d = \pm\sqrt{b}$, $c^2 = \pm 2\sqrt{b} - 2a \equiv (x_1 \pm x_3)^2$. We thus assume that neither of the expressions $\pm 2\sqrt{b} - 2a$ is the square of a rational number.

*Dickson's *Introduction to the Theory of Algebraic Equations* (Wiley & Sons), p. 56.